

# Teleparallelism, modified Born-Infeld nonlinearity and space-time as a micromorphic ether.

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## Abstract

Discussed are field-theoretic models with degrees of freedom described by the  $n$ -leg field in an  $n$ -dimensional "space-time" manifold. Lagrangians are generally-covariant and invariant under the internal group  $GL(n, \mathbf{R})$ . It is shown that the resulting field equations have some correspondence with Einstein theory and possess homogeneous vacuum solutions given by semisimple Lie group spaces or their appropriate deformations. There exists a characteristic link with the generalized Born-Infeld type nonlinearity and relativistic mechanics of structured continua. In our model signature is not introduced by hands, but is given by integration constants for certain differential equations.

**Keywords:** affinely-rigid body, Born-Infeld nonlinearity, micromorphic continuum, teleparallelism, tetrad.

## 1 Introduction

The model suggested here has several roots and arose from some very peculiar and unexpected convolution of certain ideas and physical concepts seemingly quite remote from each other. In a sense it unifies generalized Born-Infeld type nonlinearity, tetrad approaches to gravitation, Hamiltonian systems with symmetries (mainly with affine symmetry; so-called affinely-rigid body), generally-relativistic spinors and motion of generalized relativistic continua with internal degrees of freedom (relativistic micromorphic medium, a kind of self-gravitating microstructured "ether" generalizing the classical Cosserat continuum). The first two of mentioned topics (Born-Infeld, tetrad methods) were strongly contributed by Professor Jerzy Plebański, cf. e.g. [16, 18]. The same concerns spinor theory [17]. My "micromorphic ether", although in a rather very indirect way, is somehow related to the problem of motion in general relativity; the

discipline also influenced in a known way by J. Plebański [13]. There exist some links between generalized Born-Infeld nonlinearity and the modern theory of strings, membranes and p-branes. Geometrically this has to do with the theory of minimal surfaces [9].

## 2 Born-Infeld motive

Let us begin with the Born-Infeld motive of our study. No doubt, linear theories with their superposition principle are in a sense the simplest models of physical phenomena. Nevertheless, they are too poor to describe physical reality in an adequate way. They are free of the essential self-interaction. In linear electrodynamics stationary centrally symmetric solutions of field equations are singular at the symmetry centre and their total field energy is infinite. If interpreting such centres as point charges one obtains infinite electromagnetic masses, e.g. for the electron. In realistic field theories underlying elementary particle physics one usually deals with polynomial nonlinearity, e.g. the quartic structure of Lagrangians is rather typical. Solitary waves appearing in various branches of fundamental and applied physics owe their existence to various kinds of nonlinearity, very often nonalgebraic ones. General relativity is nonlinear (although quasilinear at least in the gravitational sector) and its equations are given by rational functions of field variables, although Lagrangians themselves are not rational. In Einstein theory one is faced for the first time with a very essential non-linearity which is not only nonperturbative (it is not a small nonlinear correction to some dominant linear background) but is also implied by the pre-assumed symmetry conditions, namely by the demand of general covariance. Indeed, any Lagrangian theory invariant under the group of all diffeomorphisms must be nonlinear (although, like Einstein theory may be quasilinear). Nonlinearity of non-Abelian gauge theories is also due to the assumed symmetry group. In our mechanical study of affinely-rigid bodies [20] nonlinearity of geodetic motion was also due to the assumption of invariance under the total affine group. This, by the way, established some link with the theory of integrable lattices.

The original Born-Infeld nonlinearity had a rather different background and was motivated by the mentioned problems in Maxwell electrodynamics. There was also a tempting idea to repeat the success of general relativity and derive equations of charges motion from the field equations. Unlike the problem of infinities, which was in principle solved, the success in this respect was rather limited. The reason is that in general relativity the link between field equations and equations of motion is due not only to the nonlinearity itself (which is, by the way, necessary), but first of all to Bianchi identities. The latter follow from the very special kind of nonlinearity implied by the general covariance.

As we shall see later on, some kinds of generalized Born-Infeld nonlinearities also may be related to certain symmetry demands. But for us it is more convenient to begin with some apparently more formal, geometric aspect of "Born-Infeld-ism".

In linear theories Lagrangians are built in a quadratic way from the field vari-

ables  $\Psi$ . Thus, they may involve  $\Psi\Psi$ -terms algebraically quadratic in  $\Psi$ ,  $\partial\Psi\partial\Psi$ -terms algebraically quadratic in derivatives of  $\Psi$ , and  $\Psi\partial\Psi$ -terms bilinear in  $\Psi$  and  $\partial\Psi$ ; everything with constant coefficients. In any case, the dependence on derivatives, crucial for the structure of field equations is polynomial of at most second order in  $\partial\Psi$  (linear with  $\Psi$ -coefficients in the case of fermion fields).

But there is also another, in a sense opposite pole of mathematical simplicity of Lagrangians. By its very geometrical nature, Lagrangian  $L$  is a scalar Weyl density of weight one; in an  $n$ -dimensional orientable manifold it may be represented by a differential  $n$ -form locally given by:  $\mathcal{L} = L(\Psi, \partial\Psi)dx^1 \wedge \dots \wedge dx^n$ . But as we know, there is a canonical way of constructing such densities: just taking square roots of the moduli of determinants of second-order covariant tensors,

$$L = \sqrt{|\det[L_{ij}]|}, \quad (1)$$

or rather constant multiples of this expression, when some over-all negative sign may occur. In the sequel the square-rooted tensor will be referred to as the Lagrange tensor, or tensorial Lagrangian. In general relativity and in all field theories involving metric tensor  $g$  on the space-time manifold  $M$ ,  $L$  is factorized in the following way:  $L = \Lambda(\Psi, \partial\Psi)\sqrt{|\det[g_{ij}]|}$ , where  $\Lambda$  is a scalar expression. Here the square-rooted metric  $g_{ij}$  offers the canonical scalar density. In linear theories for fields  $\Psi$  considered on a fixed metrical background  $g$ ,  $\Lambda$  is quadratic (in the aforementioned sense) in  $\Psi$ . In quasilinear Einstein theory, where in the gravitational sector  $\Psi$  is just  $g$  itself, one uses the Hilbert Lagrangian proportional to  $R[g]\sqrt{|g|}$  ( $R[g]$  denoting obviously the scalar curvature of  $g$  and  $|g|$  being an obvious abbreviation for the modulus of  $\det[g_{ij}]$ ). Obviously, Lagrangians of linear and quasilinear theories also may be written in the form (1), however, this representation is extremely artificial and inconvenient. For example, for the Hilbert Lagrangian we have  $L = \text{sign}R\sqrt{|\det[R^{2/n}g_{ij}]|}$ , i.e. locally we can write  $L_{ij} = |R|^{2/n}g_{ij}$ ;  $L_{ij} = \sqrt{|R|}g_{ij}$  if  $n = 4$ .

One can wonder whether there exist phenomena reasonably and in a convenient way described just in terms of (1). It is natural to expect that the simplest models of this type will correspond to the at most quadratic dependence of the tensor  $L_{ij}$  on field derivatives. Unlike the linear and quasilinear models, now the theory structure will be lucid just on the level of  $L_{ij}$ . Among all possible nonlinear models such ones will be at the same time quite nonperturbative but also in a sense similar to the linear and quasilinear ones.

The historical Born-Infeld model [2] is exceptional in that the Lagrange tensor is linear in field derivatives; Lagrangian is

$$L = -\sqrt{|\det[bg_{ij} + F_{ij}]|} + b^2\sqrt{|\det[g_{ij}]|}, \quad (2)$$

where  $F_{ij} = A_{j,i} - A_{i,j}$  is the electromagnetic field strength,  $A_i$  is the covector potential and the constant  $b$  is responsible for the saturation phenomenon; it determines the maximal attainable field strength. The field dynamics is encoded in the first term. The second one, independent of  $F$ , fixes the energy scale: Lagrangian and energy are to vanish when  $F$  vanishes. Therefore, up the minus

sign preceding the square root, we have  $L_{ij} = bg_{ij} + F_{ij}$ . For weak fields, e.g. far away from sources,  $L$  asymptotically corresponds with the quadratic Maxwell Lagrangian. And all singularities of Maxwell theory are removed — static spherically symmetric solutions are finite at the symmetry centre (point charge) and the electromagnetic mass is finite. The finiteness of solutions is due to the saturation effect.  $L$  has a differential singularity of the type  $\sqrt{0}$  when the field is so strong that the determinant of  $[bg_{ij} + F_{ij}]$  vanishes. Such a situation is singular-repulsive just as  $v = c$  situation for the relativistic particle, where the interaction-free Lagrangian is given by  $L = -mc^2\sqrt{1 - v^2/c^2}$  (in three-dimensional notation). The classical Born-Infeld theory is in a sense unique, exceptional among all a priori possible models of nonlinear electrodynamics [1, 18]. It is gauge invariant, the energy current is not space-like, energy is positively definite, point charges have finite electromagnetic masses and there is no birefringence. There exist solutions of the form of plane waves combined with the constant electromagnetic field; in particular, solitary solutions may be found [1].

Although the amazing success of quantum field theory and renormalization techniques (even classical ones, as developed by Dirac) for some time reduced the interest in Born-Infeld theory, nowadays this interest is again growing on the basis of new motivation connected, e.g. with strings, p-branes, alternative approaches to gravitation, etc. [5, 6, 7].

Linearity of  $L_{ij}$  in field derivatives is an exceptional feature of electrodynamics among all models developed in the Born-Infeld spirit. Usually  $L_{ij}$  must be quadratic in derivatives because of purely geometric reasons. For example, let us consider the scalar theory of light, neglecting the polarization phenomena. In linear theory one uses then the real scalar field  $\Psi$  ruled by the d'Alembert Lagrangian  $L = g^{ij}\Psi_{,i}\Psi_{,j}\sqrt{|g|}$ . The only natural "Born-Infeld-ization" of this scheme is based on

$$L = -\sqrt{|\det[bg_{ij} + \Psi_{,i}\Psi_{,j}]|} + b^2\sqrt{|\det[g_{ij}]|}, \quad (3)$$

thus, as a matter of fact  $L_{ij}$  is quadratic in field derivatives  $\partial\Psi$ . It is interesting that such a model gives for the stationary spherically symmetric solutions in Minkowski space the formula which is exactly identical with that for the scalar potential  $\varphi = A_0$  in the usual Born-Infeld model, namely, the expression  $f(r) = \sqrt{Ab} \int_0^r du/\sqrt{A + u^4}$ , where  $A$  denotes the integration constant (related to the value of point charge producing the field). Let us mention, incidentally, that such a scalar Born-Infeld model was successfully applied in certain problems of nonlinear optics. Therefore, the very use of  $L_{ij}$  quadratic in derivatives does not seem to violate philosophy underlying the Born-Infeld model. There are also other arguments. Born-Infeld theory explains point charges in a very nice way as "regular singularities", but is not well-suited to describing interactions with autonomous external sources, e.g. with the charged (complex) Klein-Gordon or Dirac fields. Combining in the usual way Born-Infeld Lagrangian with expressions describing matter fields and the mutual interactions one obtains equations involving complicated nonrational expressions. It

seems much more natural to use the expression (1) with  $L_{ij}$ , e.g. of the form:  $L_{ij} = \alpha g_{ij} + \kappa \bar{\Psi} \Psi g_{ij} + b F_{ij} + c D_i \bar{\Psi} D_j \Psi$ , where  $\alpha, \kappa, b, c$  are constants and  $D_j \Psi = \Psi_{,j} + ie A_j \Psi$  (electromagnetic covariant derivatives). Here  $\Psi$  denotes the complex scalar field and for weak fields we obtain the usual mutually coupled Maxwell-Klein-Gordon system. The same may be done obviously for field multiplets and for fermion fields. Such a model has a nice homogeneous structure and the field equations are rational in spite of the square-root expression used in Lagrangian.

Once accepting  $L_{ij}$  quadratic in derivatives we can also think about admitting such models also for the pure electromagnetic field, e.g.

$$L_{ij} = \alpha g_{ij} + \beta F_{ij} + \gamma g^{kl} F_{ik} F_{lj} + \delta g^{kr} g^{ls} F_{kl} F_{rs} g_{ij}, \quad (4)$$

where  $\alpha, \beta, \gamma, \delta$  are constants. The terms quadratic in  $F$  in (4) are known as contributions to the energy-momentum tensor of the Maxwell field.

If we try to construct Born-Infeld-like models for non-Abelian gauge fields, then the quadratic structure of  $L_{ij}$  is as unavoidable as in scalar electrodynamics, e.g.

$$L_{ij} = \alpha g_{ij} + \gamma g^{kl} F^K_{ik} F^L_{lj} h_{KL} \quad (5)$$

is the most natural expression. Obviously,  $\alpha, \gamma$  are constants,  $F^K_{ij}$  are strength of the gauge fields and  $h_{KL}$  is the Killing metric on the gauge group Lie algebra.

Lagrangians (1) may be modified by introducing "potentials", i.e. scalar  $\Psi$ -dependent multipliers at the square-root expression, or at  $L_{ij}$  itself, or finally, at the determinant. However, the less number of complicated and weakly-motivated corrections of this type, the more aesthetic and convincing is the dynamical hypothesis contained in  $L$ .

It is worthy of mentioning that the scalar Born-Infeld models with quadratic  $L_{ij}$  have to do with the theory of minimal surfaces [9] and with some interplay between general covariance and internal symmetry. Namely, we can consider scalar fields  $\Psi$  on  $M$  with values in some linear space  $W$  of dimension  $m$  higher than  $n = \dim M$ . Let  $W$  be endowed with some (pseudo)Euclidean metric  $h \in W^* \otimes W^*$ . We could as well consider pseudo-Riemannian structure as a target space, however, now we prefer to concentrate on the simplest model. If  $M$  is structureless, then the only natural possibility of constructing Lagrangian invariant under rotations  $O(W, \eta)$  and under  $\text{Diff} M$  (generally-covariant) is to take the pull-back Lagrange tensor  $L_{ij} = g_{ij} = h_{KL} \Psi^K_{,i} \Psi^L_{,j}$ . If  $h$  is Euclidean, this means that we search minimal surfaces in  $W$ ;  $M$  is used as a merely parametrization. Field equations have the form  $g^{ij} \nabla_i \nabla_j \Psi^K = 0$ ,  $K = \overline{1, m}$ , where the covariant differentiation is meant in the Levi-Civita  $g$ -sense. We can fix the coordinate gauge by putting  $((1/2)g^{ij}g^{ab} - g^{ia}g^{jb})g_{ab,i} = 0$ , e.g. making the assumption  $\Psi^i = x^i$ ,  $i = \overline{1, n}$ , i.e. identifying  $n$  of the fields  $\Psi^K$  with  $M$ -coordinates themselves. Then the gauge-free content of our field equations is given by:  $g^{ij} \Psi^\Sigma_{,ij} = 0$ ,  $\Sigma = \overline{n+1, m}$ . These equations follow from the effective Lagrange tensor

$$L^{\text{eff}}_{ij} = h_{ij} + 2h_{\Sigma(i} \Psi^\Sigma_{,j)} + h_{\Sigma\Lambda} \Psi^\Sigma_{,i} \Psi^\Lambda_{,j}. \quad (6)$$

Here we easily recognize something similar to (4), i.e. second-order polynomial in derivatives with the effective background metric  $h_{ij}$  in  $M$ . If  $h$  has the block structure with respect to  $(\Sigma, i)$ -variables, then there are no first-order terms, just as in (3),(5). It is seen that the "almost classical" Born-Infeld form with the effective metric on  $M$  may be interpreted as a gauge-free reduction of generally-covariant dynamics in  $M$  with some internal symmetries in the target space  $W$ . One can also multiply the corresponding Lagrangians by some "potentials" depending on the  $h$ -scalars built of  $\Psi$ , however with the provisos mentioned above. Surprisingly enough, such scalar models describe plenty of completely different things, e.g. soap and rubber films, geodetic curves, relativistic mechanics of point particles, strings and p-branes, minimal surfaces and Jacobi-Maupertuis variational principles. There were also alternative approaches to gravitation based on such models [14].

### 3 Tetrads, teleparallelism and internal affine symmetry

There are various reasons for using tetrads in gravitation theory, in particular for using them as gravitational potentials, in a sense more primary than the metric tensor [15]. First of all they provide local reference frames reducing the metric tensor to its Minkowskian shape. They are unavoidable when dealing with spinor fields in general relativity. This has to do with the curious fact that  $\overline{\text{GL}}^+(n, \mathbf{R})$ , the universal covering group of  $\text{GL}^+(n, \mathbf{R})$ , is not a linear group (has no faithful realization in terms of finite matrices). Also the gauge approaches to gravitation ( $\text{SL}(2, \mathbf{C})$ -gauge, Poincaré gauge models) are based on the use of tetrad fields. And even in standard Einstein theory the tetrad formulation enables one to construct first-order Lagrangians which are well-defined scalar densities of weight one. If one uses the metric field as a gravitational potential, the Hilbert Lagrangian is, modulo the cosmological term, the only possibility within the class of essentially first-order variational principles. Unlike this, the tetrad degrees of freedom admit a wide class of nonequivalent variational principles. Some of them were expected to overwhelm singularities appearing in Einstein theory.

Let us begin with introducing necessary mathematical concepts. It is convenient to consider a general "space-time" manifold  $M$  of dimension  $n$  and specify to  $n = 4$  only on some finite stage of discussion. The principal fibre bundle of linear frames will be denoted by  $\pi : FM \rightarrow M$  and its dual bundle of co-frames by  $\pi^* : F^*M \rightarrow M$ . The duality between frames and co-frames establishes the canonical diffeomorphism between  $FM$  and  $F^*M$ . The co-frame dual to  $e = (\dots, e_A, \dots)$  will be denoted by  $\tilde{e} = (\dots, e^A, \dots)$ ; by definition  $\langle e^A, e_B \rangle = \delta^A_B$ . When working in local coordinates  $x^i$  we use the obvious symbols  $e^i_A$ ,  $e^A_i$ , omitting the tilde-sign at the co-frame. Therefore,  $e^A_i e^i_B = \delta^A_B$ ,  $e^i_A e^A_j = \delta^i_j$ . The structure group  $\text{GL}(n, \mathbf{R})$  acts on  $FM$ ,  $F^*M$  in a standard way, i.e. for any  $L \in \text{GL}(n, \mathbf{R}) : e \mapsto eL = (\dots, e_A, \dots)L =$

$(\dots, e_B L^B_A, \dots)$ ,  $\tilde{e} \mapsto \tilde{e}L = (\dots, e^A, \dots)L = (\dots, L^{-1A}_B e^B, \dots)$ . Fields of (co-)frames ((co-)tetrads when  $n = 4$ ) are sufficiently smooth cross-sections of  $F^*M$ , respectively  $FM$  over  $M$ . They are affected by elements of  $GL(n, \mathbf{R})$  pointwise, according to the above rule. In gauge models of gravitation one must admit local, i.e.  $x$ -dependent action of  $GL(n, \mathbf{R})$ . Any field  $M \ni x \mapsto L(x) \in GL(n, \mathbf{R})$  acts on cross-section  $M \ni x \mapsto e_x \in FM$  according to the rule:  $(eL)_x = e_x L(x)$ . Obviously, for any  $x \in M$ ,  $e_x \in \pi^{-1}(x)$  may be identified with a linear isomorphism of  $\mathbf{R}^n$  onto the tangent space  $T_x M$ ; similarly,  $\tilde{e}_x \in \pi^{*-1}(x)$  is an  $\mathbf{R}^n$ -valued form on  $T_x M$ . Therefore, the field of co-frames is an  $\mathbf{R}^n$ -valued differential one-form on  $M$ . In certain problems it is convenient to replace  $\mathbf{R}^n$  by an abstract  $n$ -dimensional linear space  $V$ . The reason is that  $\mathbf{R}^n$  carries plenty of structures sometimes considered as canonical (e.g. the Kronecker metric) and this may lead to false ideas.

In the sequel we shall need some byproducts of the field of frames. If  $\eta$  is a pseudo-Euclidean metric on  $\mathbf{R}^n$  (on  $V$ ), then the Dirac-Einstein metric tensor on  $M$  is defined as:  $h[e, \eta] = \eta_{AB} e^A \otimes e^B$ ,  $h_{ij} = \eta_{AB} e^A_i e^B_j$ . In general relativity  $n = 4$  and  $[\eta_{AB}] = \text{diag}(1, -1, -1, -1)$ . Obviously, the prescription for  $e \mapsto h[e, \eta]$  is invariant under the local action of the pseudo-Euclidean group  $O(n, \eta)$ . In general relativity it is the Lorentz group  $O(1, 3)$  that is used as internal symmetry. The metric  $\eta$ , or rather its signature, is an absolute element of the theory.

The field of frames gives rise to the teleparallelism connection  $\Gamma_{\text{tel}}[e]$ ; it is uniquely defined by the condition  $\nabla e_A = 0$ ,  $A = \overline{1}, n$ . In terms of local coordinates:  $\Gamma^i_{jk} = e^i_A e^A_{j,k}$ . Obviously, its curvature tensor vanishes and the parallel transport of tensors consists in taking in a new point the tensor with the same anholonomic  $e$ -components. The prescription  $e \mapsto \Gamma(e)$  is globally  $GL(n, \mathbf{R})$ -invariant,  $\Gamma[eA] = \Gamma[e]$ ,  $A \in GL(n, \mathbf{R})$ . The torsion tensor of  $\Gamma_{\text{tel}}$ ,  $S[e]^i_{jk} = \Gamma_{\text{tel}}^i_{[jk]} = (1/2)e^i_A (e^A_{j,k} - e^A_{k,j})$  may be interpreted as an invariant tensorial derivative of the field of frames. It is directly related to the non-holonomy object  $\gamma$  of  $e$ ,  $S^i_{jk} = \gamma^A_{BC} e^i_A e^B_j e^C_k$ ,  $[e_A, e_B] = \gamma^C_{AB} e_C$  (as usual,  $[u, v]$  denotes the Lie bracket of vector fields  $u, v$ ).

In general relativity tetrad field is interpreted as a gravitational potential; the space-time metric  $h[e, \eta]$  is a secondary quantity. When expressed through  $e$ , Hilbert Lagrangian may be invariantly reduced to some well-defined scalar density of weight one and explicitly free of second derivatives. Indeed, one can show that

$$L_H = R[h[e]]\sqrt{|h|} = (J_1 + 2J_2 - 4J_3)\sqrt{|h|} + 4(S^a_{ab}h^{bi}\sqrt{|h|})_{,i}, \quad (7)$$

where  $|h|$  is an abbreviation for  $|\det[h[e]_{ij}]|$  and  $J_1 = h_{ai}h^{bj}h^{ck}S^a_{bc}S^i_{jk}$ ,  $J_2 = h^{ij}S^a_{ib}S^b_{ja}$ ,  $J_3 = h^{ij}S^a_{ai}S^b_{bj}$  are Weitzenböck invariants built quadratically of  $S$ . They are invariant under the global action of  $O(1, 3)$  on  $e$ . The last term in (7) is a well-defined scalar density and the divergence of some vector density of weight one. It absorbs the second derivatives of  $e$ . Therefore, Hilbert Lagrangian is equivalent to the first term in (7),

$$L_{H-\text{tel}} := L_1 + 2L_2 - 4L_3 = (J_1 + 2J_2 - 4J_3)\sqrt{|h|}. \quad (8)$$

It is invariant under the local action of  $O(1, 3)$  modulo appropriate divergence corrections. And resulting field equations for  $e$  are exactly Einstein equations with  $h[e, \eta]$  substituted for the metric tensor. In this sense one is dealing with different formulation of the same theory. Obviously, due to the mentioned local  $O(1, 3)$ -invariance, the tetrad formulation involves more gauge variables.

The use of tetrads as fundamental fields opens the possibility of formulating more general dynamical models. The simplest modification consists in admitting general coefficients at three terms of (8),

$$L = c_1 L_1 + c_2 L_2 + c_3 L_3. \quad (9)$$

When the ratio  $c_1 : c_2 : c_3$  is different than  $1 : 2 : (-4)$ , the resulting model loses the local  $O(1, 3)$ -invariance and is invariant only under the global action of  $O(1, 3)$ . The whole tetrad  $e$  becomes a dynamical variable, whereas in (8) everything that does not contribute to  $h[e, \eta]$  is a pure gauge. Models based on (9) were in fact studied and it turned out that in a certain range of coefficients  $c_1, c_2, c_3$  their predictions agree with those of Einstein theory and with experiment. One can consider even more general models with Lagrangians non-quadratic in  $S$ :

$$L(S, h) = g(S, h) \sqrt{|h|}, \quad (10)$$

where  $g$  is arbitrary scalar intrinsically built of  $S, h$ , e.g. some nonlinear function of Weitzenböck invariants. The resulting theories are not quasilinear any longer. There were some hopes to avoid certain non-desirable infinities by appropriate choice of  $g$  (people were then afraid of singularities, nowadays they love them). To write down field equations in a concise form it is convenient to introduce two auxiliary quantities  $H_i{}^{jk} := \partial L / \partial S^i{}_{jk} = e^A{}_i H_A{}^{jk} = e^A{}_i \partial L / \partial e^A{}_{j,k}$ ,  $Q^{ij} := \partial L / \partial h_{ij}$  referred to, respectively, as a field momentum and Dirac-Einstein stress. They are tensor densities of weight one. One can show that equations of motion have the form:  $K_i{}^j := \nabla_k H_i{}^{jk} + 2S^l{}_{ik} H_l{}^{jk} - 2h_{ik} Q^{kj} = 0$ . The covariant differentiation is meant here in the  $e$ -teleparallelism sense.

To the best of our knowledge, all teleparallelism models of gravitation belonged to the above described class. They are invariant under global action of  $O(n, \eta)$  (i.e.  $O(1, 3)$  in the physical four-dimensional case). Let us observe, however, that there are some fundamental philosophical objections concerning this symmetry. The corresponding local symmetry in Einstein theory was well-motivated. It simply reflected the fact that the tetrad field was a merely non-holonomic reference frame, something without a direct dynamical meaning. It was only its metrical aspect  $h[e, \eta]$  that was physically interpretable. If we once decide seriously to make the total  $e$  a dynamical quantity, the global  $O(e, \eta)$ -symmetry evokes some doubts. Why not the total  $GL(n, \mathbf{R})$ -symmetry? Why to introduce by hands the Minkowskian metric  $\eta$  to  $\mathbf{R}^n$ , the internal space of tetrad field? Such questions become very natural when, as mentioned above, we use an abstract linear space  $V$  instead of  $\mathbf{R}^n$ . From the purely kinematical point of view the most natural group is  $GL(V)$ . It seems rather elegant to use a bare, amorphous linear space  $V$  than to endow it a priori with geometrically nonmotivated absolute element  $\eta \in V^* \otimes V^*$ . To summarize: when one gives



up the local Lorentz symmetry  $O(V, \eta)$ , then it seems more natural to use the global  $GL(V)$  than global  $O(V, \eta)$ . Then  $L$  in (10) does not depend on  $h$  and our field equations for Lagrangians  $L(S)$  have the following general form:

$$K_i^j := \nabla_k H_i^{jk} + 2S^l{}_{lk} H_i^{jk} = 0. \quad (11)$$

If the model is to be generally-covariant, that we always assume, then some Bianchi-type identities imply that  $L$  is an  $n$ -th order homogeneous function of  $S$ ,  $S^i{}_{jk} \partial L / \partial S^i{}_{jk} = S^i{}_{jk} H_i^{jk} = nL$ , i.e.  $L(\lambda S) = \lambda^n L(S)$  for any  $\lambda > 0$ . This is a kind of generalized Finsler structure.

If we search for models with internal linear-conformal symmetry  $\mathbf{R}^+ O(V, \eta) = e^{\mathbf{R}} O(V, \eta)$ , then  $L$  must be homogeneous of degrees 0 in  $h$ ,  $h_{ij} \partial L / \partial h_{ij} = h_{ij} Q^{ij} = 0$ , i.e.  $L(S, \lambda h) = L(S, h)$  for any  $\lambda \in \mathbf{R}^+$ .

The simplest  $GL(n, \mathbf{R})$ -invariant ( $GL(V)$ -invariant) and generally-covariant models have the following generalized Born-Infeld structure:  $L = \sqrt{|\det[L_{ij}]|}$  with the Lagrange tensor quadratic in derivatives:

$$L_{ij} = 4\lambda S^k{}_{im} S^m{}_{jk} + 4\mu S^k{}_{ik} S^m{}_{jm} + 4\nu S^k{}_{lk} S^l{}_{ij}, \quad (12)$$

where  $\lambda, \mu, \nu$  are real constants. One can in principle complicate them and make more general multiplying the above Lagrangian  $L$ , or Lagrange tensor  $L_{ij}$ , or the under-root expression (to some extent the same procedure) by a function of some basic  $GL(V)$ -invariant and generally-covariant scalars built of  $S$ . All such scalars are zeroth-order homogeneous functions of  $S$ .

The first two terms of (12) are symmetric and may be considered as a candidate for the metric tensor of  $M$  built of  $e$  in a globally  $GL(V)$ -invariant and generally-covariant way:

$$g_{ij} = \lambda \gamma_{ij} + \mu \gamma_i \gamma_j = 4\lambda S^k{}_{im} S^m{}_{jk} + 4\mu S^k{}_{ik} S^m{}_{jm}. \quad (13)$$

The best candidate is the dominant term  $\gamma_{ij}$  built of  $S$  according to the Killing prescription.

The mentioned scalar potentials used for multiplying (12) may be built of expressions like  $\gamma_{il} \gamma^{jm} \gamma^{kn} S^i{}_{jk} S^l{}_{mn}$ ,  $\gamma^{ij} S^k{}_{ik} S^m{}_{jm}$ ,  $\Gamma^i{}_j \Gamma^j{}_k \dots \Gamma^l{}_m \Gamma^m{}_i$ , etc., where  $\Gamma_{ij} := 4S^k{}_{lk} S^l{}_{ij}$ ,  $\Gamma^i{}_j = \gamma^{im} \Gamma_{mj}$  and  $\gamma^{ij}$  is reciprocal to  $\gamma_{ij}$ ,  $\gamma^{ik} \gamma_{kj} = \delta^i{}_j$ ; we assume it does exist.

No doubts, the simplest and maximally "Born-Infeld-like" are models without such scalar potential terms, with the Lagrange tensor (12) quadratic in derivatives.

Due to the very strong nonlinearity, it would be very difficult to perform in all details the Dirac analysis of constraints resulting from the Lagrangian singularity. Nevertheless, the primary and secondary constraints may be explicitly found. If the problem is formulated in  $n$  dimensions and all indices both holonomic and nonholonomic are written in the convention  $K, i = \overline{0, n-1}$  (zeroth variable referring to "time"), then primary constraints, just as in electrodynamics, are given by  $\pi^0{}_K = 0$ , where  $\pi^i{}_K$  are densities of canonical momenta conjugated to "potentials"  $e^K{}_i$ . Thus, there are a priori  $n$  redundant

variables among  $n^2$  quantities  $e^i_K$  and they may be fixed by coordinate conditions, like, e.g.  $e^i_K = \delta^i_K$  for some fixed value of  $K$  or  $e^{K,i} = e^{K,j}g^{ji} = 0$  (Lorentz transversality condition). Secondary constraints are related to the field equations free of second "time" derivatives, these may be shown to be:  $K_i^0 = \nabla_j H_i^{0j} + 2S^k_{kj} H_i^{0j} - 2h_{ij} Q^{j0} = 0$ . We have left the  $Q$ -term, because the statement, just as that about primary constraints is valid both for affinely-invariant and Lorentz invariant models. Obviously, for affine models the  $Q$ -term vanishes. Let us observe an interesting similarity to the empty-space Einstein equations, where secondary constraints are related to  $R_i^0 = 0$ . Similarly, for the free electromagnetic field:  $H^{0j}_{,j} = \text{div } \vec{D} = 0$ .

As mentioned, discussion of the consistency of our model in terms of Dirac algorithm would be extremely difficult. Nevertheless, one can show that our field equations are not self-contradictory (this might easily happen in models invariant under infinite-dimensional groups with elements labelled by arbitrary functions). Namely, one can explicitly construct some particular solutions of very interesting geometric structure. Of course, there is still an unsolved problem "how large" is the general solution.

Analysing the structure of equations (11) one can easily prove the following

**Theorem 1** *If field of frames  $e$  has the property that its "legs"  $e_A$  span a semi-simple Lie algebra in the Lie-bracket sense,  $[e_A, e_B] = \gamma^C_{AB} e_C$ ,  $\gamma^C_{AB} = \text{const}$ ,  $\det[\gamma^C_{DA} \gamma^D_{CB}] \neq 0$ , then  $e$  is a solution of (11) for any  $GL(n, \mathbf{R})$ -invariant model of  $L$ , in particular, for (12).*

Roughly speaking, this means that semisimple Lie groups, or rather their group spaces are solutions of variational  $GL(n, \mathbf{R})$ -invariant field equations for linear frames. They are homogeneous, physically non-excited vacuums of the corresponding model. Fixing some point  $a \in M$  we turn  $M$  into semisimple Lie group. Its neutral element is just  $a$  itself,  $e_A$  generate left regular translations and are right-invariant. This gives rise also to left-invariant vector fields  ${}^a e_A$  generating right regular translations,  $[e_A, e_B] = \gamma^C_{AB} e_C$ ,  $[{}^a e_A, {}^a e_B] = -\gamma^C_{AB} {}^a e_C$ ,  $[e_A, {}^a e_B] = 0$ . The tensor  $\gamma_{ij}$  becomes then the usual Killing metric on Lie group; it is parallel with respect to the teleparallelism connection  $\Gamma_{\text{tel}}[e]$ ,  $\nabla \gamma_{ij} = 0$ . This means that  $(M, \gamma, \Gamma_{\text{tel}})$  is a Riemann-Cartan space. For the general  $e$  it is not the case. For semisimple Lie-algebraic solutions there exists such a bilinear form  $\eta$  on  $\mathbf{R}^n$  (on  $V$ ) that:  $\gamma[e] = h[e, \eta]$  and obviously  $\eta_{AB} = \gamma^C_{DA} \gamma^D_{CB}$ . The metric field  $\gamma[e]$  has  $2n$  Killing vectors  $e_A, {}^a e_A$ . Obviously, within the  $GL(n, \mathbf{R})$ -invariant framework  $\gamma[e]$  is a more natural candidate for the space-time metric than  $h[e, \eta]$ . In the special case of Lie-algebraic frames they in a sense coincide, but neither  $\eta$  itself nor even its signature are a priori fixed. Instead, they are some features, a kind of integration constants of some particular solutions.

Let us mention, there is an idea according to which all fundamental physical fields should be described by differential forms (these objects may be invariantly differentiated in any amorphous manifold [8, 21, 22, 23]). Of particular interest are the special solutions, constant in the sense that their differentials are

expressed by constant-coefficients combinations of exterior products of primary fields.

The question arises as to a possible link between the above  $\text{GL}(n, \mathbf{R})$ -framework and the ideas of general relativity. For Lie-algebraic fields of frames some kind of relationship does exist. Namely, if  $\gamma_{ij}$  is the Killing metric on a semisimple  $n$ -dimensional Lie group,  $R_{ij}$  is its Ricci tensor and  $R$  — the curvature scalar, then, as one can show [12]

$$R_{ij} - \frac{1}{2}R\gamma_{ij} = -\frac{1}{8}(n-2)\gamma_{ij}. \quad (14)$$

Rescaling the definition of the metric tensor on  $M$ ,  $g_{ij} = a\gamma_{ij}$ ,  $a = \text{const}$ , we obtain  $R_{ij} - (1/2)Rg_{ij} = \Lambda g_{ij}$ ,  $\Lambda = -(n-2)/8a$ , and these are just Einstein equations with a kind of cosmological term. Therefore, at least in a neighbourhood of group-like vacuums the both models seem to be somehow interrelated.

There is however one disappointing feature of affinely-invariant  $n$ -leg models and their interesting and surprising Lie group solutions. Everything is beautiful for the abstract, non-specified  $n$ . But for our space-time  $n = 4$  and there exist no semisimple Lie algebras in this dimension. There are, fortunately, a few supporting arguments:

1. We can try to save everything on the level of Kaluza-Klein universes of dimension  $n > 4$ . It is interesting that the  $n$ -leg field offers the possibility of deriving the very fibration of such a universe over the usual four-dimensional space-time as something dynamical, not absolute as in Kaluza-Klein theory. The fibration and the structural group would then appear as features of some particular solutions.
2. One can show that Lie-algebraic solutions exist in some sense for systems consisting of the  $n$ -leg and of some matter field, e.g. the complex scalar field  $\Psi$ . Lagrange tensor is then given by  $L_{ij} = (1 - a\bar{\Psi}\Psi)\gamma_{ij} + b\bar{\Psi}_{,i}\Psi_{,j}$ . Even if  $e_A$  span a nonsimple Lie algebra, there are  $(e, \Psi)$ -solutions with  $\det[L_{ij}] \neq 0$  and with the oscillating complex unimodular factor at  $\Psi$  [10, 11]. The same may be done for higher-dimensional multiplets of matter fields,  $L_{ij} = (1 - a_{kl}^-\bar{\Psi}^k\Psi^l)\gamma_{ij} + b_{kl}^-\bar{\Psi}^k_{,i}\Psi^l_{,j}$ .
3. In dimensions "semisimple plus one" (e.g. 4) there exist also some geometric solutions with the group-theoretical background. They are deformed trivial central extensions of semisimple Lie groups [21].

Let us describe roughly the last point. We fix some Lie-algebraic  $n$ -leg field  $E = (\dots, E_A, \dots) = (E_0, \dots, E_\Sigma, \dots)$ , where  $A = \overline{0, n-1}$ ,  $\Sigma = \overline{1, n-1}$ , and the basic Lie brackets are as follows:  $[E_0, E_\Sigma] = 0$ ,  $[E_\Sigma, E_\Lambda] = E_\Lambda C^\Delta_{\Sigma\Lambda}$ , and  $\det[C_{\Lambda\Gamma}] := \det[C^\Sigma_{\Lambda\Delta} C^\Delta_{\Gamma\Sigma}] \neq 0$ . In adapted coordinates  $(\tau, x^\mu) = (x^0, x^\mu)$  (where  $\mu = \overline{1, n-1}$ ) we have  $E_0 = \partial/\partial\tau$ ,  $E_\Sigma = E^\mu_\Sigma(x)\partial/\partial x^\mu$ . The dual coframe  $E = (\dots, E^A, \dots) = (E^0, \dots, E^\Sigma, \dots)$  is locally represented as:  $E^0 = d\tau$ ,  $E^\Sigma = E^\Sigma_\mu(x)dx^\mu$ ,  $E^\Sigma_\mu E^\mu_\Lambda = \delta^\Sigma_\Lambda$ . The corresponding Lie algebra obviously is not semisimple. But we can construct new fields of frames  $e$  or  $'e$  given

respectively by  $e = \rho E$ ,  $'e_0 = E_0$ ,  $'e_\Sigma = \rho E_\Sigma = e_\Sigma$ , where  $\rho$  is a scalar function such that  $e_\Sigma \rho = E_\Sigma \rho = 0$ , i.e. in adapted coordinates it depends only on  $\tau$ ,  $\partial \rho / \partial x^\mu = 0$ .

**Theorem 2** *For any  $\rho$  without critical points, both  $e$  and  $'e$  are solutions of any  $GL(n, \mathbf{R})$ -invariant and generally covariant equations (11). In both cases  $\gamma[e] = \gamma['e]$  is stationary and static in spite of the expanding (contracting) behaviour of  $e$ ,  $'e$ .*

If the Lie algebra spanned by  $(E_1, \dots, E_{n-1})$  is of the compact type, then  $\gamma[e]$  is normal-hyperbolic and has the signature  $(+ - \dots -)$  with respect to the nonholonomic basis  $(E_0, \dots, E_\Sigma, \dots)$ , thus the  $\tau$ -variable and coordinates  $x^i$  have respectively time-like and space-like character. The above function  $\rho$  is a purely gauge variable and in appropriately adapted coordinates:

$$\gamma[e] = \gamma['e] = dx^0 \otimes dx^0 + {}_{(n-1)}\gamma_{\alpha\beta}(x^\kappa) dx^\alpha \otimes dx^\beta, \quad (15)$$

where  $x^0 := \pm \sqrt{(n-1)} \ln(x^0/\delta)$ ,  $\delta$  is constant,  ${}_{(n-1)}\gamma_{\alpha\beta} = 4S^\kappa{}_{\lambda\alpha} S^\lambda{}_{\kappa\beta}$ . Obviously, in all formulas the capital and small Greek indices, both free and summed run over the "spatial" range  $\overline{1, n-1}$  (conversely as in the usually used notation).

Another, coordinate-free expression:  $\gamma[e] = (n-1) (d\rho/d\tau)^2 e^0 \otimes e^0 + \rho^2 C_{\Lambda\Sigma} e^\Lambda \otimes e^\Sigma$ . With such solutions  $M$  becomes locally  $\mathbf{R}_{\text{time}} \times G_{\text{space}}$ ,  $G$  denoting the  $(n-1)$ -dimensional Lie group with structure constants  $C^\Delta_{\Lambda\Sigma}$ . The above metric  $\gamma$  has  $(2n-1)$  Killing vectors; one time-like and  $2(n-1)$  space-like ones, when  $G$  is compact-type. This is explicitly seen from the formula (15), or its coordinate-free form  $\gamma = (n-1) (d\ln\rho/d\tau)^2 E^0 \otimes E^0 + C_{\Lambda\Sigma} E^\Lambda \otimes E^\Sigma$ . If we introduce spinor fields, then in their matter Lagrangians we must use the Dirac-Einstein metric  $h[e, \eta]$  with  $\eta$  of the form:  $\eta_{00} = \beta = \text{const}$ ,  $\eta_{0\Lambda} = 0$ ,  $\eta_{\Lambda\Sigma} = C_{\Lambda\Sigma}$ . This metric is subject to the cosmological expansion (contraction) known from general relativity, e.g.  $h['e, \eta]$  in its spatial part expands according to the de Sitter rule. Therefore, in spite of stationary-static character of  $\gamma$ , the test spinor matter will witness about cosmological expansion (contraction). This may be an alternative explanation of this phenomenon. If  $n = 4$  there are the following Lie-algebraic-expanding vacuum solutions:  $\mathbf{R} \times \text{SU}(2)$  or  $\mathbf{R} \times \text{SO}(3, \mathbf{R})$  with the normal-hyperbolic signature  $(+ - - -)$ , the plus sign related to  $E_0$ . There are also solutions of the form  $\mathbf{R} \times \text{SL}(2, \mathbf{R})$ ,  $\mathbf{R} \times \text{SL}(2, \mathbf{R})$ ; they have the signature  $(+ + + -)$ ; now the time-like contribution has to do with the "compact dimension" of  $\text{SL}(2, \mathbf{R})$ , whereas the mentioned "expansion" holds in one of spatial directions. It is seen that our  $\text{GL}(n, \mathbf{R})$ -models in a sense distinguish both the normal-hyperbolic signature and the dimension  $n = 4$ , just on the basis of solutions of local differential equations. In any case, something like the  $\eta$ -signature of standard tetrad description is not here introduced by hands.

Finally, let us observe that one can speculate also about another cosmological aspects of our model. In generally-relativistic spinor theory one uses the Dirac amplitude, tetrad and spinor connection (or affine Einstein-Cartan connection) as basic dynamical variables. The corresponding matter (Dirac) Lagrangian is locally  $\text{SO}(1, 3)$ - or rather  $\text{SL}(2, \mathbf{C})$ -invariant. The same concerns

gravitational Lagrangian for the tetrad and spinor connection either in Einstein or in gauge-Poincaré form. The idea was formulated some time ago that the true gravitational Lagrangian should contain a term which is only globally invariant under internal symmetries. Additional tetrad degrees of freedom were then expected to have something to do with the dark matter, at least in a part of it [3, 4]. Our  $GL(4, \mathbf{R})$ -models would be from this point of view optimal.

Finally, let us notice that our "expanding" Lie solutions for dimensions "semisimple plus one" might be cosmologically interpreted as the motion of cosmical relativistic fluid ( $e_0$ -legs of the tetrad) with internal affine degrees of freedom ( $e_\Sigma$ -legs). This would be something like the relativistic micromorphic continuum [20, 21].

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